Gravity waves over a non-uniform flow

By H. G. M. VELTHUIZEN AND
L. VAN WIJNGAARDEN

Twente Institute of Technology, Enschede, The Netherlands

This paper is concerned with the propagation of small amplitude gravity waves over a flow with non-uniform velocity distribution. For such a flow Burns derived a relation for the velocity of propagation in terms of the velocity distribution of the mean flow. This result is derived here in another way and some of its implications are discussed. It is shown that one of these is hardly acceptable physically. Burns's result holds only when a real value of the propagation velocity is assumed; the mentioned difficulties vanish if complex values are allowed for, implying damping or growth of the waves. Viscous effects which are the cause of damping or growth are important in the wall layer near the bottom and also at the critical depth, which is present when the wave speed is between zero and the fluid velocity at the free surface.

In §2 the basic equations for the present problem are given. In §3 exchange of momentum and energy between wave and primary flow is discussed. This is analogous to what happens at the critical height in a wind flow over wind-driven gravity waves. In §4 the viscous effects at the bottom are included in the analysis and the complex equation for the propagation velocity is derived. Finally in §5 illustrations of the theory are given for long waves over running flow and for the flow along a ship advancing in a wavy sea. In these examples a negative curvature of the mean velocity profile is shown to have a stabilizing effect.

1. Introduction

A familiar result from inviscid hydrodynamics is (see for example, Lamb 1932, §233) the dispersion relation

$$c^2 = \frac{g}{k} \tanh kh \tag{1.1}$$

for gravity waves of small amplitude travelling on the free surface of an otherwise undisturbed fluid of depth h. In (1.1) c is the phase velocity of the wave and k the wave-number, while g is the acceleration due to gravity. When the fluid moves with uniform velocity U_0 the same relation holds in a frame of reference in which the undisturbed fluid is at rest, whence the wave speed c follows from

$$(U_0 - c)^2 = g/k \tanh kh. \tag{1.2}$$

This paper is concerned with the form of the dispersion equation when the velocity distribution is non-uniform between the bottom and the free surface.

52

In (two-dimensional) wave flow let the bottom be at $y = 0$ and the free surface at $y = h$, gravity pointing in negative y direction, the wave travelling in the x direction. For long waves, i.e. $kh \ll 1$, with which we shall primarily be concerned, an estimate for the phase velocity can be made as follows. In a frame moving with the wave velocity c the motion is steady. Hence along a streamline the total head is constant. We denote the mean flow by $U(y)$, the wave-induced velocities in the horizontal and vertical directions by u and v, the pressure by p and the fluid density by ρ. The constancy of the total head means that

$$p + \tfrac{1}{2}\rho\{(U - c + u)^2 + v^2\} + \rho gy = \text{constant} \tag{1.3}$$

along a streamline. In the theory of long waves on fluids at rest, vertical acceleration is neglected which results in a hydrostatic pressure distribution in the vertical direction. We make this assumption here also and assume further that the wave amplitude is small enough to permit the neglect of squares and products of wave-induced velocities. The expression (1.3) reduces to

$$(U - c)\, u + g\eta = 0, \tag{1.4}$$

where $\eta(x, t)$ is the wave elevation above the mean height h. In linear conservative waves there is equipartition between energies, averaged over a wavelength. In this case, assuming a conservative wave, the kinetic energy and the potential energy must, averaged over a wavelength λ, be equal. Upon introducing

$$\eta = a\, e^{ikx} \tag{1.5}$$

the potential energy in a wavelength is $\tfrac{1}{4}\rho ga^2\lambda$. Hence

$$\tfrac{1}{4}\rho ga^2\lambda = \tfrac{1}{2}\rho \int_0^\lambda dx \int_0^h u^2\, dy, \tag{1.6}$$

where the vertical wave-induced velocity is neglected with respect to the horizontal one, since we are dealing with long waves. From (1.4) we have

$$u = -\frac{ga \exp ikx}{U - c}.$$

Inserting this in (1.6) yields

$$\int_0^h \frac{dy}{(U - c)^2} = g^{-1}. \tag{1.7}$$

This relation has been previously obtained by Burns (1953) in a quite different way, to be discussed in the following sections. For constant U, U_0 say, (1.7) reduces to the long wave approximation of (1.2). For a general distribution of U also two values of c are found from (1.7). Consider a profile varying monotonically from U_1† at the bottom to U_h at the free surface. A striking result from (1.7), discussed by Burns, is that c is either larger than U_h or smaller than U_1, because otherwise U would equal c somewhere, which makes the integral divergent. For $U_1 = 0$ this means that also for $U_h > (gh)^{\frac{1}{2}}$ there is the possibility of up-stream propagation. This is hardly acceptable physically. Consider, for instance, a distribution as sketched in figure 1. For high Froude number F_r, defined by

$$F_r = U_h^2/gh, \tag{1.8}$$

† In a non-viscous approximation U may be non-zero.

the two possible values of c are with uniform flow, $U = U_h$, somewhat larger or less respectively than U_h.

According to (1.7) the introduction of shear over a very small part of the flow as in figure 1, would drastically change this because one of the values of c is negative. Burns, in discussing the consequences of (1.7) raised some doubts on this conclusion and suggested that viscous effects have to be taken into account.
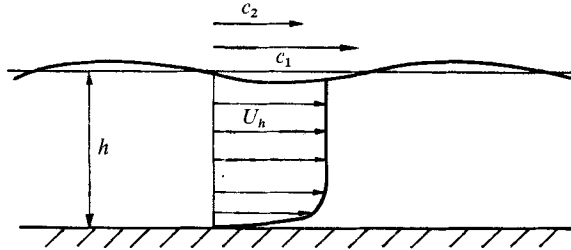

FIGURE 1. Waves over flow sheared near the bottom, $F_r \gg 1$. With uniform flow $U = U_h$, the possible velocities of propagation are c_1 and c_2. In the sheared flow one of the velocities points upstream according to (1.7).

Here we examine this problem further. We notice that Burns's conclusion ceases to be valid when complex values of c are admitted,

$$c = c_r + ic_i. \tag{1.9}$$

This implies damping or growth of the waves, which may be caused by viscous effects. These are of importance in the region near the bottom, where $U = 0$, but also in the region where U equals c_r, if such a region is present. The governing equations with pertinent boundary conditions are derived in §2. As will be shown in §3, the divergence of the integral in (1.7) is closely related to the well known singularity of the inviscid Orr–Sommerfeld (or Rayleigh) equation. In the next sections the corrections due to viscous effects are discussed.

2. Governing equations

The wave-induced velocities u and v satisfy the condition of mass conservation when we introduce the stream function Ψ defined by

$$u = \frac{\partial \Psi}{\partial y}, \quad v = -\frac{\partial \Psi}{\partial x}. \tag{2.1}$$

The linearized Navier–Stokes equations are, taking into account that the primary flow also satisfies the Navier–Stokes equations,

$$\frac{\partial u}{\partial t} + U \frac{\partial u}{\partial x} + v \frac{dU}{dy} = \nu \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) - \frac{1}{\rho} \frac{\partial p}{\partial x}, \tag{2.2}$$

$$\frac{\partial v}{\partial t} + U \frac{\partial v}{\partial x} = \nu \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right) - \frac{1}{\rho} \frac{\partial p}{\partial y} - g. \tag{2.3}$$

We consider waves travelling in the x direction with velocity c and therefore put

$$\Psi = \phi(y)\, e^{ik(x-ct)}. \tag{2.4}$$

Introduction of (2.1) and (2.4) in (2.2) and (2.3) results in the well known Orr–Sommerfeld equation

$$(U-c)\,\phi'' - \{k^2(U-c)+U''\}\phi = -\frac{i\nu}{k}(\phi^{iv} - 2k^2\phi'' + k^4\phi), \tag{2.5}$$

where primes denote differentiation with respect to y.

The velocity profile $U(y)$ used here is laminar because from the primary flow only the mean value is taken into account and not the fluctuations, present when the flow is turbulent. Strictly speaking, results of the present theory apply to laminar flow only.

Application to turbulent flow, under the assumption that the distribution of mean flow is the most important for the effects under investigation, is questionable to the same extent as the use of a 'pseudo laminar' flow by Miles (1957) in his theory of wave generation by a turbulent wind flow. The conditions to be satisfied by ϕ are partly bottom conditions, partly free surface conditions. At the bottom u and v must vanish, or from (2.1) and (2.4)

$$y = 0, \quad \phi = 0, \tag{2.6}$$

$$y = 0, \quad \phi' = 0. \tag{2.7}$$

Equation (2.5) is the fundamental equation in the theory of stability of laminar flow, where flow between solid boundaries is usually considered. Then on all boundaries $\phi = \phi' = 0$. Here we have as complementary conditions those at the free surface. The kinematical condition is, in linearized form,

$$\left(\frac{\partial}{\partial t} + U_h \frac{\partial}{\partial x} + v \frac{\partial}{\partial y}\right)(y-\eta) = 0 \quad \text{at} \quad y = h.$$

Using (1.5), (2.1) and (2.4) this becomes

$$y = h: \quad (U_h - c)\, a = -\phi. \tag{2.8}$$

The other condition is on the pressure. At the free surface, $y = h + \eta$, this can be written as
$$p_{y=h+\eta} = p_{y=h} + (\partial p/\partial y)_{y=h}\,\eta + \ldots.$$

Because at the free surface $p = 0$ and because v is of the first order in the wave amplitude a, we find, inserting $\partial p/\partial y$ from (2.3), that up till the second order in the wave amplitude
$$p = \rho g \eta \quad \text{at} \quad y = h. \tag{2.9}$$

Note that (2.9) does not involve, as Burns (1953) states, the long wave approximation of a hydrostatic pressure distribution throughout the fluid. For fluids of small viscosity the viscous terms in (2.2) and (2.3) may be neglected at the free surface, where they are of higher order in an appropriately defined reciprocal Reynolds number then at the bottom. With the neglect of the viscous terms, substitution of (2.9) into (2.2) gives, using again (1.5), (2.1) and (2.4)

$$y = h: \quad (U_h - c)\, \phi' - \phi U' = -ga. \tag{2.10}$$

Equation (2.5), together with the boundary conditions (2.6)–(2.8) and (2.10), determine the problem mathematically.

Viscous effects will be important near solid boundaries. Far from these the viscous terms in (2.5) may be neglected. The Orr–Sommerfeld equation then reduces to

$$(U-c)\,\phi'' - \{k^2(U-c)+U''\}\,\phi = 0, \tag{2.11}$$

sometimes called the Rayleigh equation.

In the limit of vanishing c_i this equation has a singularity where $U = c_r$. The implications of this critical layer for the present problem are discussed in the next section.

3. Critical layer

From the theory of hydrodynamic stability and in particular from the work by Miles (1957) and Benjamin (1959) on wave generation by wind, we know how to deal with the critical layer where $U = c_r$. For long waves, $kh \ll 1$, the Rayleigh equation (2.11) has the solution, satisfying (2.8) and (2.10)

$$\phi = -a(U-c)\left\{1 - g\int_y^h \frac{dy}{(U-c)^2}\right\}. \tag{3.1}$$

In the limit of vanishing imaginary part of c, $c_i \to 0$, this solution is valid only for $y \geqslant y_c$, y_c being given by $U(y_c) = c_r$.

From the aforementioned work we know that for $y < y_c$, but far from solid boundaries, the proper way to cope with the singularity of the integrand in (3.1) is to encircle it in the integration with a path *below* the singularity, which yields

$$y < y_c: \phi = -a(U-c)\left\{1 - \fint_y^h \frac{g\,dy}{(U-c)^2} + \frac{ig\pi U_c''}{U_c'^3}\right\}. \tag{3.2}$$

The stroke through the integral in (3.2) means that the principal value is meant. This cannot be obtained from the integral as it stands. For numerical calculations the integral has, by partial integration, to be transformed into a more suitable form (see appendix).

Burns, apart from neglecting the viscous effects at $y = 0$, imposed the 'inviscid' boundary condition $\phi = 0$ on (3.1), overlooking apparently the fact that this solution is, when a critical layer occurs, only valid above this layer and not at $y = 0$.

As is well known, the sign of the imaginary term in (3.2) is indeterminate when only the Rayleigh equation is considered. The way in which this sign is prescribed by the full equation is treated, e.g. in Lin's (1955) book on hydrodynamic stability, where it is also explained that the effects at the critical layer give rise to a difference in wave-induced Reynolds stress above and below $y = y_c$. Denoting averaging over a wavelength with a bar, this Reynolds stress is, averaged over a wavelength, $-\rho\overline{uv}$. The relation for this difference in the Reynolds stress is (Lin 1955) for small c_i

$$(-\rho\overline{uv})_{y>y_c} - (-\rho\overline{uv})_{y<y_c} = \frac{\pi\rho k}{2}\frac{U_c''}{U_c'}|\phi_c|^2\exp(2kc_it). \tag{3.3}$$

This relation is a cornerstone in the theory of wave generation by wind. When y is measured from the free surface upwards, it is known there that for large y the Reynolds stress vanishes because both u and v vanish for large y. It follows that a Reynolds stress exists for $y < y_c$ with the sign of $-U_c''$ for $U_c' > 0$. In the present problem such a conclusion cannot be drawn directly because there is no region where the Reynolds stress is *a priori* known. At the bottom, of course, the Reynolds stress is zero but in the wall layer near the bottom a stress can be built up.

The Reynolds stress can be expressed in terms of ϕ and its complex conjugated $\check{\phi}$ by (cf. Lin 1955, § 8.2)

$$-\rho\overline{uv} = \tfrac{1}{4}ik\rho(\phi\check{\phi}' - \check{\phi}\phi')\exp(2kc_i t). \qquad (3.4)$$

At the free surface this yields with help of the free surface conditions (2.8) and (2.10) and using (1.9)

$$y = h: \ -\rho\overline{uv} = \frac{k\rho a^2 g c_i}{U_h - c_r}\exp(2kc_i t). \qquad (3.5)$$

This Reynolds stress at the surface changes the wave momentum, which is in the direction of the positive x axis given by

$$M = -\frac{1}{2}\frac{\rho g a^2 \lambda}{U_h - c_r}\exp(2kc_i t), \qquad (3.6)$$

in a wavelength.

The rate of change of wave momentum, in the linear approximation from which (3.6) results concentrated between crests and troughs, equals minus the Reynolds stress at the surface (see figure 2), or

$$(-\rho\overline{uv})_{y=h} = -dM/dt, \qquad (3.6a)$$

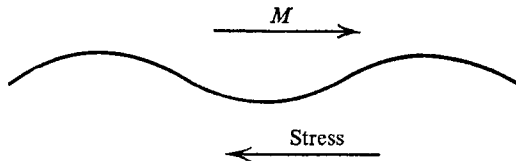confirmed by comparison of (3.5) and (3.6).

FIGURE 2. Illustration of (3.6a); a positive Reynolds stress tends to decrease a positive wave momentum.

Whether the momentum decreases or increases depends on the sign of c_i, which is determined by including the wall layer in our considerations.

So far we have used the inviscid Orr–Sommerfeld equation. Only the sign of the imaginary term in (3.2) is based on considerations which include viscous effects. It is known that for $c_i < 0$ the thickness of the critical layer does not reduce to zero in the limit of zero viscosity. The expression (3.1) for $c_i < 0$ is not valid, even for infinite Reynolds number, at $y = y_c$, nor (3.2) right below $y = y_c$. The jump of the horizontal velocity ϕ' across the critical layer is according to

(3.1) and (3.2) given by $i\pi U_c'' g a / U_c'^2$, which can, since it follows from (3.1) that $\phi_c = g a / U_c'$, also be written as $i\pi U_c'' / U_c' \phi_c$.

Although the inviscid solution fails in the critical layer for $c_i < 0$ the difference in ϕ' is also given there by the same expression, as follows from numerical calculations by Betchov & Criminale (1967, §10). Since ϕ itself is continuous we may use (3.3) also for $c_i < 0$. The exact behaviour of ϕ through the critical layer is of no importance here. When $c_i > 0$, for which we shall discuss an example in §5, the behaviour of ϕ is accurately described by (3.1) and (3.2) in the limit of small c_i. In order to obtain the dispersion equation for c we proceed to incorporate the viscous effects at the bottom in the expression for ϕ in order to impose on this expression the bottom conditions (2.6) and (2.7).

4. Viscous effects in wall region

We follow the classic pattern in this type of problem and construct a solution of (2.5) consisting of the inviscid solution discussed in the foregoing section and one of the so called viscous solutions. First for convenience we make the variables dimensionless by putting

$$y_1 = y/h, \tag{4.1}$$

$$u_1 = u/U_h. \tag{4.2}$$

We introduce the Reynolds number

$$R = U_h h / \nu, \tag{4.3}$$

and the dimensionless wave-number

$$\alpha = kh. \tag{4.4}$$

For long waves α is a small quantity. At the wall U'' is negligibly small, so that a function $f*$ that satisfies

$$f*^{iv} - iR\alpha(U_1 - c_1)f*'' = 0, \tag{4.5}$$

is an approximate solution for large R of the full Orr–Sommerfeld equation, from which terms in (4.5) are the dominant ones provided U_1'' and α are small. A further simplification is to approximate the velocity profile in the wall region by a linear one. This approximation has been discussed by Benjamin (1959) who showed that this approximation, good in laminar flow, is a reasonable one also for turbulent flow. In the wall layer we write, suppressing the subscripts of the dimensionless variables, $$U = yU_0'. \tag{4.6}$$

Inserting this in (4.5) gives the 'viscous' equation in the wall layer

$$f*^{iv} - iR\alpha(yU_0' - c)f*'' = 0. \tag{4.7}$$

This equation has four solutions. Two of these, being solutions of $f*'' = 0$, are non-viscous and of no interest here. The remaining ones are rapidly varying viscous solutions. The pertinent one is the solution which tends to zero for large y. We denote this solution by f and consider

$$\phi = -a(U - c)\left\{1 - F_r^{-1} \int_y^1 \frac{dy}{(U - c)^2}\right\} + Bf, \tag{4.8}$$

where B is a constant. The right-hand side of (4.8) is an approximate solution for large R of (2.5), valid near the wall $y = 0$. Because c is complex the existence of a value of y such that $U(y_c) = c_r$ does not introduce a singularity in the integrand in (4.8). Evaluating the integral assuming a small value of c_i gives rise to a singularity and then the integral has to be split up into its principal value and the contribution of the singularity. In §3 we have done this to bring about the connexion with related results from the theories of hydrodynamic stability and wave generation by wind. Without specifying that c_i is small the integral in (4.8) is convergent as it stands. On (4.8) we impress the boundary conditions for ϕ and ϕ' given by (2.6) and (2.7). Elimination of B gives the required equation for c

$$F_r - \int_0^1 \frac{dy}{(U-c)^2} = \frac{1}{c\left\{U_0' + c\dfrac{f'(0)}{f(0)}\right\}}. \tag{4.9}$$

The quotient $f'(0)/f(0)$ can be expressed in the Tietjens function of argument $(\alpha R U_0')^{\frac{1}{3}} c/U_0'$ by

$$\frac{f'(0)}{f(0)} = -\frac{U_0'}{c} \frac{1}{F(\sigma c/U_0')}. \tag{4.10}$$

Here

$$\sigma = (\alpha R U_0')^{\frac{1}{3}}, \tag{4.10a}$$

and F is the Tietjens function (see for a definition Miles 1960). There is some advantage in using, instead of $F(z)$,

$$\mathscr{F}(z) = (1 - F(z))^{-1}. \tag{4.11}$$

Then (4.9) can be written as

$$F_r - \left\{\int_0^1 \frac{dy}{(U-c)^2}\right\} = \frac{1}{U_0' c}\left\{1 - \mathscr{F}\left(\frac{\sigma c}{U_0'}\right)\right\}. \tag{4.12}$$

Relation (4.12) is the principal result of this investigation. To obtain some further insight we assume $|c_i/c_r| \ll 1$. Then the integral in (4.12) reduces to

$$\int_0^1 \frac{dy}{(U-c_r)^2} - \frac{\pi i U_c''}{U_c'^3} + 2ic_i \int_0^1 \frac{dy}{(U-c_r)^3}.$$

Upon decomposing $\mathscr{F}$ into its real and imaginary parts,

$$\mathscr{F} = \mathscr{F}_r + i\mathscr{F}_i,$$

we obtain collecting real and imaginary terms and neglecting terms of order $(c_i/c_r)^2$,

$$F_r - \int_0^1 \frac{dy}{(U-c_r)^2} = \frac{1}{U_0' c_r}\{1 - \mathscr{F}_r(\Delta)\}, \tag{4.13}$$

$$c_i = \frac{\dfrac{\pi U_c''}{U_c'^3} + \dfrac{\mathscr{F}_i(\Delta)}{U_0' c_r}}{2\displaystyle\int_0^1 \frac{dy}{(U-c_r)^3} + \frac{1}{U_0' c_r^2}\{\mathscr{F}_r(\Delta) - \Delta\mathscr{F}_r'(\Delta) - 1\}}. \tag{4.14}$$

In these expressions $\Delta = \sigma c_r/U_0'$. Tables of the real part $\mathscr{F}_r(z)$ and the imaginary part $\mathscr{F}_i(z)$ of $\mathscr{F}(z)$ are given by Miles (1960).

It would be interesting to discuss the stability of the laminar flow $U(y)$, bounded on one side by a rigid wall and on the other side by a free surface, on the basis of (4.13) and (4.14). This, however, is beyond the scope of the present paper and will be left for future research. Here we restrict ourselves to illustrating by some examples the gross features of (4.13) and (4.14).
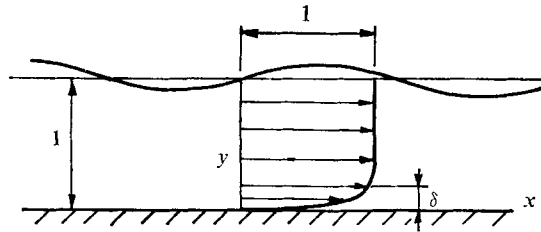

FIGURE 3. Gravity wave over velocity distribution, which is uniform with unit velocity, except for region of dimensionless thickness δ.

5. Some examples

(a) First we consider a flow as in figure 3. The mean flow consists of a uniform flow over most of the height and a boundary-layer type of flow over a region of extent δ. Therefore U_0' is of order h/δ and may be assumed to be large. For a particular profile the integral in (4.13) can be integrated numerically (see appendix). For an estimate we write

$$\int_0^1 \frac{dy}{(U-c_r)^2} = \frac{1}{(1-c_r)^2} + \int_0^1 2\frac{yU'\,dy}{(U-c_r)^3}.$$

The second integral is small, the integrand being zero near $y = 0$ as well as near $y = 1$. Its value is of order $(U_0')^{-1}$.

Inserting this in (4.13) shows that

$$F_r = \frac{1}{(1-c_r)^2} + O\left(\frac{1}{U_0'}\right). \tag{5.1}$$

This shows that the value of the propagation velocity is mainly determined by the velocity at the free surface.

The result (5.1) does not lead to the difficulties mentioned in §1, and arising from Burns's result (1.7). It follows from (5.1) that there is a value of c_r between 0 and 1 for $F_r > 1$. In that case there is a critical depth, contributing to c_i. For the profile sketched in figure 3, $U'' < 0$. The sign of c_i depends on the signs of the various terms in (4.14). The general case is rather complicated but some simplifying assumptions are in order here. From (5.1) we have to order $(U_0')^{-1}$

$$c_r = 1 - F_r^{-\frac{1}{2}}. \tag{5.2}$$

For not too large values of F_r (but of course exceeding unity) $U - c_r$ is positive over most of the flow and

$$\int_0^1 \frac{dy}{(U-c_r)^3}$$

may be assumed to be positive and of order 1.

From Miles's (1960) discussion on $\mathscr{F}$ it follows that the expression between curly brackets in (4.14) tends to -1 for $\Delta \to 0$ and to 0 for $\Delta \to \infty$, in between never exceeding unit order. Because this expression is preceded by $(U_0')^{-1}$ its neglect is reasonable for moderate values of F_r.

Then

$$2c_i \!\!\fint_0^1 \frac{dy}{(U-c_r)^3} \sim \frac{\pi U_c''}{U_c'^3} + \frac{\mathscr{F}_i(\Delta)}{U_0' c_r}.$$

The right-hand side summarizes the contributions to c_i by the critical layer, which is negative here, and by the wall layer. It is interesting to note that the effect of the critical layer is, with $U_c'' < 0$, to stabilize the wave, whereas in the wave generation by wind a negative curvature in the wind profile at the critical height has a destabilizing effect.

Consider the profile $U = 1 - \exp(-yu_0')$. For the profile to be linear in the wall layer it is necessary that $(\alpha R U_0')^{-\frac{1}{3}}$, which is the thickness of the wall layer, is small with respect to $(U_0')^{-1}$. This implies $\Delta \gg 1$.

Then $\mathscr{F}_i$ is negligibly small (Miles 1960). For this profile the integral in (4.12) can easily be evaluated and we find†

$$c_i \sim -\frac{\pi}{2U_0' F_r^{\frac{1}{2}}}. \tag{5.3}$$

The question naturally arises whether this effect is measurable under laboratory conditions. In an experiment one would try to measure decay of the wave amplitude in distance, not in time. For small c_i this can be inferred from the timewise behaviour by replacing t by x/c_r in the non-periodic part of the elevation. The wave elevation is given by $\eta = a\exp\{i(kx - ct)\}$ and accordingly, in an experiment in which the waves decay or grow in distance

$$\frac{1}{\eta_{\max}} \frac{d\eta_{\max}}{dx} = \frac{kc_i}{c_r}.$$

Using (5.2) and (5.3) we find for the relative decay per wavelength

$$\frac{\Delta\eta_{\max}}{\eta_{\max}} \sim -\frac{\pi^2}{U_0'(F_r^{\frac{1}{2}} - 1)}.$$

For a depth h and a boundary-layer thickness δ, U_0' is of order h/δ. Taking $h/\delta = O(10^2)$, e.g. $h \sim 0{\cdot}1$ m and $\delta \sim 10^{-3}$ m, it follows that an appreciable change in the wave height must occur in, say, 10 wavelengths.

Over this distance the mean profile will change. Therefore, in trying out the theory in an experiment the change of the mean profile must be taken into account.

(b) As a second example consider a ship advancing in a wavy sea of infinite depth with velocity V (figure 4). In the boundary layer alongside the ship fluid is entrained. If we make a cross-section A–A in the boundary layer parallel to

† The argument of $\ln(-c/(1-c))$ occurring in the evaluation is determined by assuming c_i to be positive. This follows from the fact that the pertinent solution of the inviscid Orr–Sommerfeld equation for $c_i \to 0$ is that one for which c_i tends to zero from the positive side.

the ship's side the distribution of the velocity component parallel to V will be somewhat like that sketched in figure 5, the maximum velocity being of order V. The interaction of the ship's motion with the incoming waves may in various ways lead to the radiation of energy away from the ship. Here we focus attention on the interaction, in terms of the mechanism described in the foregoing sections, between the boundary layer alongside the ship and waves propagating in the direction of V. Whereas in the foregoing sections long waves were discussed, we
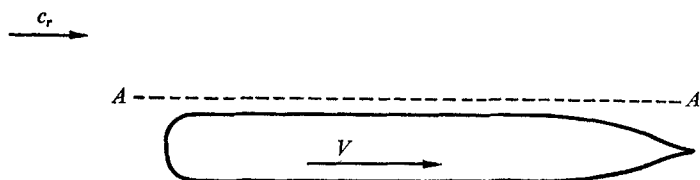

FIGURE 4. Ship advancing in wavy sea with velocity V. Waves propagating in direction of V have velocity $c_r < V$. The cross section $A-A$ is in the boundary layer along the ship.
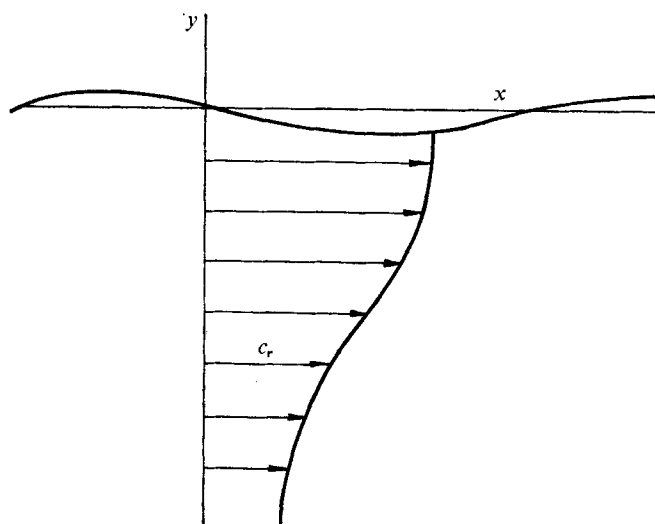

FIGURE 5. Sketch of distribution of velocity component parallel to V as it will appear in the cross-section $A-A$ of figure 4. A critical depth for which $U'' > 0$ is indicated.

have to deal here with deep water waves. An estimate for the propagation speed of such waves on a non-uniform flow may be obtained as follows. At the surface $U' = 0$ when a light fluid, e.g. air, is above the surface. Therefore U' is small near the surface and then the approximate solution of (2.11)

$$\phi = -a(U-c)e^{ky} \tag{5.4}$$

may be used, where for convenience the origin is shifted to the free surface. Solutions of this type have been used by Miles (1957) and Benjamin (1959). Equation (5.4) satisfies (2.6), vanishes for $y \to -\infty$ and is an approximate solution of (2.11) when either of the quantities $U'/k(U-c)$ and kU'/U'' is small. On (5.4)

we impress the remaining surface condition (2.10) and obtain the dispersion relation

$$(U_h - c)^2 = g/k. \tag{5.5}$$

Using this estimate of wave speed it follows that when $V = 10\,\mathrm{m/s}$ there is for waves with a wavelength between 0 and 60 m a critical depth.

At a number of these, in particular those associated with the shorter waves, U_c'' is positive (figure 5). For infinite depth the wave-induced Reynolds stress vanishes at great depth and then it follows from (3.3) that there is above the critical depth a positive stress, which according to (3.6) and (3.6a) leads to an increase of the absolute value of the wave momentum. Hence there is a transfer of momentum and energy from the boundary layer to the waves, which manifests itself as an increase in wave resistance for the ship. The amount of transmitted energy depends on the magnitude of $(U''/U')\,|\phi|^2$ at the critical depth.

For long water waves we could obtain the value for ϕ directly. For deep water, $kh \gg 1$, the analysis must be extended. Since there are no bottom effects, use of the Rayleigh equation is justified, provided of course the necessary measures regarding the critical depth are taken. Without carrying out the analysis, which is more complicated than for long waves, it is possible to give a lower and an upper bound for ϕ_c, the value of ϕ at the critical depth, for deep water. From integration of (2.11) we obtain with help of the free surface condition (2.10)

$$(U - c)\,\phi' - \phi U' = -ga + \int_0^y k^2(U - c)\,\phi\,dy. \tag{5.6}$$

At the critical depth $U = c$, whence

$$\phi_c = \frac{ga}{U_c'} - \frac{1}{U_c'}\int_0^{y_c} k^2(U - c)\,\phi\,dy. \tag{5.7}$$

For the evaluation of the integral in (5.7) we use the approximation (5.4) for ϕ. Although this is certainly not a good approximation for ϕ in the neighbourhood of the critical depth this introduces only a small error since the integrand is small anyway in the neighbourhood of the critical depth because of the factor $U - c$. Inserting (5.4) in (5.7) yields

$$\phi_c = \frac{ga}{U_c'}\left\{1 + \frac{1}{g}\int_0^{y_c} k^2(U - c)^2\,e^{ky}\,dy\right\}.$$

In the region between critical depth, $y = y_c$, and free surface, $y = 0$, we have $0 < (U - c)^2 < (U_h - c)^2$.

Introducing this in the above integral and taking (5.5) into account gives as a lower bound for ϕ_c

$$\phi_c > \frac{ga}{U_c'}e^{ky_c}.$$

This shows that, other quantities being equal, the effect of the critical depth is in deep water smaller by a factor e^{ky_c} as compared with shallow water. For three-dimensional boundary layers as occurring along a ship no velocity profiles are known quantitatively. It seems reasonable to assume that the region of appreciable curvature and gradient will occur at a depth of order of the draft of the ship. Waves having an associated critical depth of this order of magnitude and a length exceeding the draft in magnitude, will contribute to the transfer of energy from

the mean flow into the waves. In this way interaction between the waves and the viscous boundary layer is produced.

Joosen (1966) investigated the increase of wave resistance of a ship in waves on the basis of potential theory, taking no account of the boundary-layer effects. He compared his theory with experimental results and found that the agreement was good at wavelengths of order of magnitude of the length of the ship (the wave fronts being normal to the ship's course).

A discrepancy appeared at short wavelengths, because there the theoretically derived extra wave resistance tends to zero in contrast with experiments. Apart from the diffraction effect, mentioned by Joosen, the mechanism described here may contribute to the 'added resistance' of the ship in waves.

A short version of this paper was presented by one of us (L.v.W.) at the Twelfth International Congress of Applied Mechanics at Stanford University, August 1968.

Appendix

In the paper the principal value integral

$$I = \int_0^1 \frac{dy}{(U-c_r)^2},$$

appears. If one wants to evaluate the integral numerically for a given profile one will find the above form unsuitable, because the integrand is even around the point where $U = c_r$.

A form suitable for numerical calculation can be obtained in the following way. By partial integration

$$I = -\int_0^1 \frac{U''}{U'^2}\frac{dy}{U-c_r} - \left\{\frac{1}{U'}\frac{1}{U-c_r}\right\}\Big|_0^1.$$

The integral is now a Cauchy principal value, but both the first and the second term diverge at $y = 1$, where $U' = 0$. Therefore we write

$$\left(\frac{1}{U'}\right)_{y=1} = -\int_0^1 \frac{U''}{U'^2}\,dy + \frac{1}{U'_0},$$

and obtain
$$I = -\frac{1}{U'_0(1-c_r)c_r} + \frac{1}{1-c_r}\int_0^1 \frac{U''}{U'^2}\frac{U-1}{U-c_r}\,dy.$$

The integral is now convergent and a Cauchy principal value integral.

REFERENCES

BENJAMIN, T. B. 1959 *J. Fluid Mech.* **6**, 161.
BETCHOV, R. & CRIMINALE, W. O. 1967 *Stability of Parallel Flows*. New York: Academic.
BURNS, J. C. 1953 *Proc. Camb. Phil. Soc.* **49**, 695.
JOOSEN, W. P. A. 1966 *Proc. 6th Symposium on Naval Hydrodynamics*. Ed. R. D. Cooper and S. W. Doroff, Washington D.C.
LAMB, H. 1932 *Hydrodynamics*, 6th ed. Cambridge University Press.
LIN, C. C. 1955 *The Theory of Hydrodynamic Stability*. Cambridge University Press.
MILES, J. W. 1957 *J. Fluid Mech.* **3**, 185.
MILES, J. W. 1960 *J. Fluid Mech.* **8**, 593.